

Spectral Analysis and Allan Variance Calculation in the case of Phase Noise

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Abstract

In this work, time series analysis techniques are used to analyze sequential, equi-spaced mass measurements of a Si density artifact, collected from an electromechanical transducer. Specifically, techniques such as Power Spectral Density, Bretthorst periodogram, Allan variance and Modified Allan variance can provide much insight regarding the stochastic correlations that are induced on the outcome of an experiment by the measurement system and establish criteria for the limited use of the classical variance in metrology. These techniques are used in conjunction with power law models of stochastic noise in order to characterize time or frequency regimes by pointing out the different types of frequency modulated (FM) or phase modulated (PM) noise. In the case of phase noise, only Modified Allan variance can tell between white PM and flicker PM noise. Oscillations in the system can be detected accurately with the Bretthorst periodogram. Through the detection of colored noise, which is expected to appear in almost all electronic devices, a lower threshold of measurement uncertainty is obtained and the white noise model of statistical independence can no longer provide accurate results for the examined data set.

Key words: phase noise, time-series analysis, Fourier analysis, Bretthorst method, Allan variance

1 Introduction

Finding correlations in time series of measurements and defining the type of stochastic noise apparent in the measured signal remain very crucial issues for modern metrology. According to Zhang, when measurements are autocorrelated, the use of classical variance for the estimation of type A uncertainty of a data set, leads to wrong results [1]. Moreover, power law models of the type $1/f^a$ [2] are necessary to describe the type of stochastic noise that characterizes the system, which produces the time series.

In most cases, due to colored noise generation, the Frequency Modulated White Noise model (White-FM) fails to describe the system. An additional problem is Phase Modulated Noise (PM), apparent in all electrical oscillators [3] and more complex devices, as thoroughly reviewed and analyzed by Rubiola [4]. PM Noise is attributed to the phase fluctuations (or frequency fluctuations) of a signal produced by a real, non-ideal oscillator. In this oscillator the frequency of oscillation is modulated by electronic noise, inevitably leading to random frequency fluctuations in time and thus affecting the performance of the system that uses the oscillator. A list of reasons for the PM Noise generation is given by Howe [5] but still its origin remains a controversy issue in the areas of metrology.

Generally there have been utilized several methods from both time and frequency domain, in order to best analyze the response of metrological instruments, electromechanical transducers in our case study. In this paper we will focus on Allan and Modified Allan variance (time domain) as well as Schuster and Bretthorst periodograms (frequency domain).

Allan variance, initially introduced by Allan in time and frequency metrology [6], is an alternative approach for the variance calculation of autocorrelated measurements and has been thoroughly used by Witt in the area of electrical measurements [7]. Because of the fact that this method cannot tell the difference between White PM Noise and Flicker PM Noise, the Modified Allan variance has been used instead [8].

In the frequency domain, Fast Fourier Transform and Schuster periodograms remain the most widely used tools for evaluating the different types of noise in the signal's Power Spectrum. Schuster periodograms can detect oscillations in the signal and interpret them as individual peaks. Due to the limitations and in certain occasions misleading results of the Discrete Fourier Transform, an alternative method is proposed, based on Bayesian analysis: the Bretthorst periodogram [9]. This method also identifies oscillations in a signal and can distinguish between two very close frequency peaks that FFT treats as one.

The data on which the above methods will be tested, was obtained from characterization experiments of a newly commissioned 1kg/ 10mg resolution mass comparator at the Hellenic Institute of Metrology density laboratory. The principle function of this comparator will be the measurement of mass in air of Si and ceramic density artifacts of mass ca. 1kg in the form of spheres through comparative weighing. It should be noted that these methods can be applied to any electromechanical transducer system equally well with similar results.

2 General Considerations

Let's consider an input signal of the following simple sinusoidal form

$$V_{in}(t) = V_c \sin(2\pi f_c t) \quad (1)$$

The output of a non-ideal oscillator will be of the form

$$V_{out}(t) = (V_c + \varepsilon(t)) \sin(2\pi f_c t + \phi(t)) \quad (2)$$

where the time variations in amplitude are included into $\varepsilon(t)$ and the frequency deviation from the fundamental one is described by $\varphi(t)$, called the phase fluctuations of the output signal. Ignoring amplitude variations and changing to frequency regime, phase fluctuations with a spectral density $S_\phi(f)$ and frequency fluctuations with a spectral density $S_y(f)$, are described with equations (3) and (4), with h_i representing the intensity coefficients of each power law noise contribution and α, β are the spectral exponents:

$$S_y(f) = \sum_{a=-2}^{+2} h_a f^a \quad (3)$$

$$S_\phi(f) = \sum_{\beta=-4}^0 h_\beta f^\beta, \beta = a - 2 \quad (4)$$

3 Computational Methods

3.1 Fourier Spectral Analysis

The Discrete Fourier Transform is one of the most powerful tools in signal analysis and thoroughly used in many physical systems. The periodogram was introduced by Schuster, as a method of detecting periodicities and estimating their frequencies and is essentially the squared magnitude of the discrete fourier transform of the data. In 1965, Cooley and Tukey introduced the Fast Fourier Transform, a method that led to more efficient computer calculations and is considered nowadays an optimal frequency and power spectral estimator.

By plotting $S(f) - f$ in a log-log diagram, the different noise levels in the power spectrum can be estimated from the adjustment of equation (3) to the log-log plot. The calculation of the spectral exponent α , leads to the identification of white noise ($\alpha=0$), flicker noise ($\alpha=-1$), random walk noise ($\alpha=-2$) or any intermediate case $-2 < \alpha < 2$ of colored noise [7].

Unfortunately, the Discrete Fourier Transform will provide accurate frequency estimates only when the following conditions are met [9,10]: 1) the length of the data series N is sufficiently large, 2) the data series is stationary, 3) there is no evidence of low frequency existence, 4) the data contain one stationary frequency, 5) the noise is white. When one or more of the above conditions is violated (the last one in our work) an alternative method is proposed, based on Bayesian Analysis, called the Bretthorst Periodogram.

3.2 Bretthorst Periodogram

Bayesian Probability Theory defines a probability of occurrence as a reasonable degree of belief, given some prior information of the data. In our example the probability distribution of a frequency of oscillation ω is computed conditional on the data D and the prior information I , abbreviated as the posterior probability $P(\omega|D,I)$. To calculate this one must find the direct probability (or likelihood function) $P(D|\omega,I)$, the prior probability $P(\omega|I)$, the probability $P(D|I)$ (in parameter estimation problems it is a simple normalization constant) and eliminate the nuisance parameters.

$$P(\omega|D, I) = \frac{P(D|\omega, I) \cdot P(\omega|I)}{P(D|I)} \quad (5)$$

Based simply on Bayes product and sum rules [11] one can obtain the posterior probability of the frequency with or without prior knowledge of the noise variance σ^2 in the signal, concluding to equations (6a) and (6b) correspondingly. Specifically (6b) is the well-known t-student distribution. An analytical proof of these relations can be found in [9, 10]. $C(\omega)$ is the Schuster periodogram and \bar{d}^2 is the observed mean-square data value.

$$P(\omega|D, \sigma, I) \propto \exp(C(\omega)/\sigma^2) \quad (a) \quad P(\omega|D, I) \propto [1 - \frac{2C(\omega)}{Nd^2}] \quad (b) \quad (6)$$

Basically, the above equations give evidence that a single stationary frequency is present in the data and because of the processing in equations (6a-6b), all details of the periodogram are suppressed except the frequency peak, observed as a single spike in the Bretthorst periodogram.

3.3 Allan Variance

The Allan variance or Two-sample variance was first introduced by David W. Allan for the evaluation of the stability of time and frequency standards [6]. As well established, the most common measure of dispersion is the classical variance, whose value decreases as the number of data points included in the calculations, increases. Unfortunately this is only true in the case of truly random processes, where the variance of the mean decreases with the number N of data points as $\text{var}(X) = \frac{\sigma^2}{N}$.

Allan stated that in the case of autocorrelated data the above equation does not apply since there exists a possibility that the estimated variance will diverge as the number of data points increases [12]. So, we create a high-pass filter by extracting each $k+1$ value from its previous one k and thus we remove any possible trends, fast fluctuations or other peculiar characteristics. The Allan variance $\sigma_y^2(\tau)$ is estimated at time intervals $\tau = m\tau_0$, where τ_0 is a minimum sampling time and m is usually chosen to denote powers of two.

$$\sigma_y^2(\tau) = \frac{1}{2(N-1)} \sum_{k=1}^{N-1} (y_{k+1}(\tau) - y_k(\tau))^2 \quad (7)$$

From the resulting plot of $\sigma_y^2(\tau)$ - τ one can estimate the cut-off value after which the inclusion of more data points does not lead to the decrease of the variance. In a log-log plot the Allan variance is proportional to τ^μ and $\mu = -\alpha - 1$, where α is the spectral exponent appearing in equation (3). Special cases [6,7] are: $\mu=1$ (random walk noise), $\mu=0$ (flicker noise) and $\mu=-1$ (white noise).

3.4 Modified Allan Variance

This variance was first introduced in the domain of optics because it divides white phase noise from flicker phase noise by the different dependence on gate time. More specifically, ModVar is proportional to τ^{-3} for the former and τ^{-2} for the latter and ModStd is proportional to $\tau^{-3/2}$ and τ^{-1} correspondingly [13]. Given an infinite sequence $\{x_k\}$ of samples evenly spaced in time with sampling period τ_0 , the ModVar is defined as

$$Mod\sigma_y^2(\tau) = \frac{1}{2n^4\tau_0^2(N-3n+1)} \sum_{j=1}^{N-3n+1} \left[\sum_{i=j}^{n+j-1} (x_{i+2n}(\tau) - 2x_{i+n}(\tau) + x_i(\tau))^2 \right] \quad (8)$$

where $n=1,2,\dots,N/3$. The MVar obeys to a power law of the observation time τ of the form $Mod\sigma_y^2 \sim k\tau^\mu$ where $\mu = -3 - a$ [14].

4 Results

After removing the deterministic component (linear trend because of the ambient temperature) of the data with the application of a first difference filter, the resulting series is carefully examined for correlations and noise. The lag plot and autocorrelation function of the first-differenced time series are presented below, in fig. (1) and fig. (2) correspondingly.

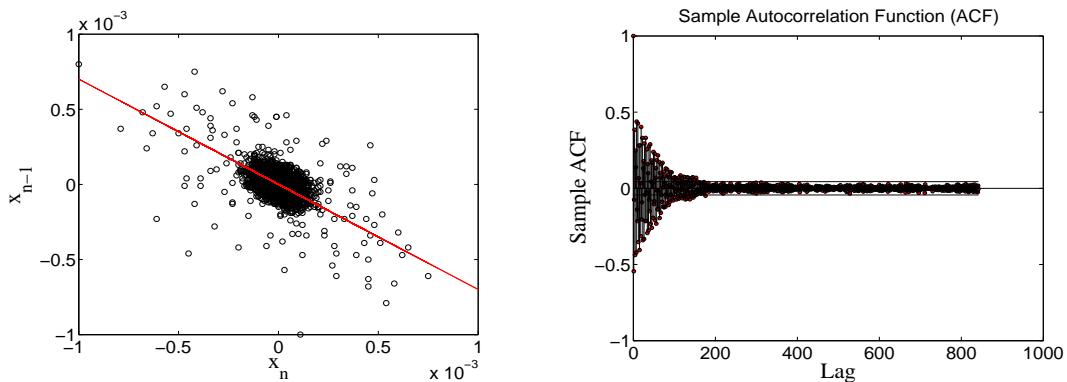


Figure 1. Lag Plot with negative correlations.

Figure 2. Autocorrelation function with oscillatory behavior.

The lag plot reveals negative correlations between successive data points with lag=1 but is unable to tell between White and Flicker PM Noise. Moreover, the autocorrelation function changes between positive and negative values for the first 200 lags (strong correlations) and afterwards, the data become statistically independent falling well within the 95% confidence bands of $\pm \frac{2}{\sqrt{N}}$, implying that the white noise model is now sufficient to describe the measurements. In order to determine safely whether or not to use Allan Variance, additional information is needed from the frequency domain analysis. The Schuster periodogram and the Power Spectrum of the data are shown in the following figures.

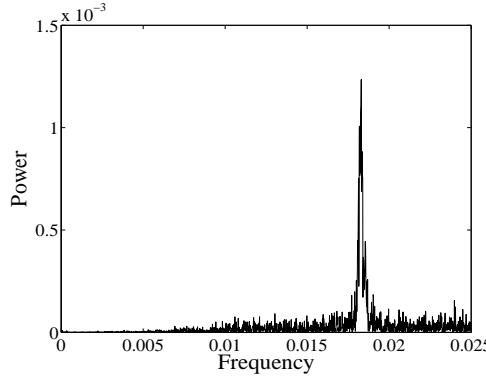


Figure 3. Periodogram with a single peak.

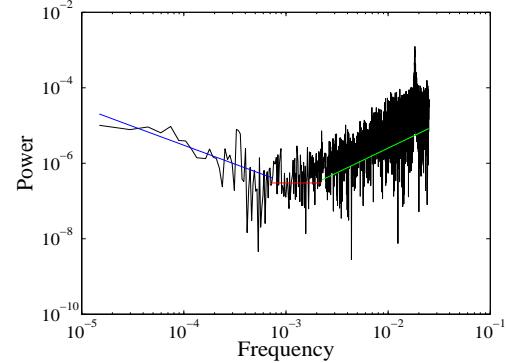


Figure 4. Power Spectrum with three different noise regions.

The periodogram reveals a strong oscillation at approximately 0.0183 Hz (or 54.64 sec). The power spectrum in a log-log scale, seems to be divided in three regions: at lower frequencies the blue line indicates a power law model of the form $1/f$ (Flicker FM Noise), the red line indicates that the White FM Noise model is adequate to describe the data and finally at higher frequencies the green line indicates a power law model $1/f^{-1}$ or $1/f^{-2}$ (PM Noise). Thus, there exist two cut-off frequencies at which evident changes in slope occur, revealing the different noise areas. The total power spectrum is given by the equation $S_y(f) = h_0 f_0 + h_{-1} f_{-1} + h_1 f_1$ (or $h_2 f_2$), where h_i are the intensity coefficients. In order to be sure for the oscillation that FFT revealed, we apply the Bretthorst methodology and according to theory we expect a single spike at the frequency of 0.0183 Hz. In order to keep the same quantities of measurement we will calculate $P(f|D,I)$. As can be observed in fig. (5), the single spike (red line) confirms the existence of $f_{osc} = 0.0183$ Hz and as predicted by theory all the other information of the periodogram (black diagram) is suppressed to a uniform plateau of zero slope.

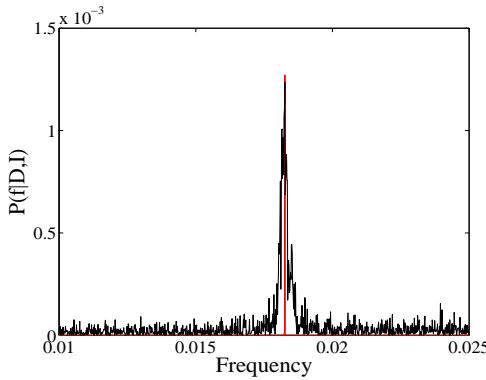


Figure 5. Comparison of Schuster and Bretthorst periodogram

From all the above, it is obvious that the use of classical variance for the calculation of type A uncertainty is inadequate and Allan variance will be used instead. In addition, Modified Allan variance will give a straightforward answer for the phase noise characterization problem in our data set. Fig. (6) below, gives the dependence of the Allan deviation over time. Clearly, there exist three different regions, in accordance with the power spectrum of fig. (4): the phase noise region $\mu=-1$ (or $\mu=-2$ for the Allan Variance) indicated by the blue line, the white FM noise area $\mu=-0.5$ (or $\mu=-1$ for the Allan variance) adjusted by the red line and finally the flicker FM floor ($\mu=0$). For $\tau=256$, $\sigma(\tau)=1.9*10^{-6}$ in contrast to classical std which estimates $\sigma=9.57*10^{-5}$. Note that the last two points are not included in the calculation. Probable causes of the flicker floor are power supply voltage fluctuations, changes in the components of the artifact and microwave power changes. Fig. (7) gives the dependence of the Modified Allan deviation over time. The slope of the blue fitted line is $-3/2$, which indicates that white PM Noise appears at small times and correspondingly high frequencies, as already observed in the power spectrum.

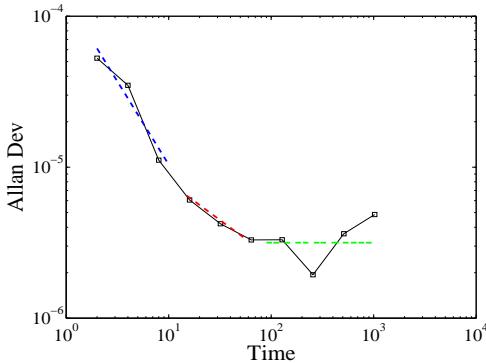


Figure 6. Allan variance.

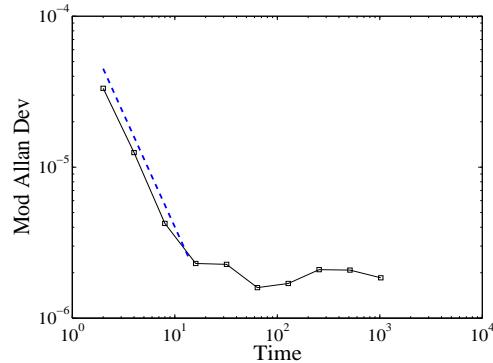


Figure 7. Modified Allan variance.

5 Conclusions

The main purpose of this work is to point out the importance of considering measurement results as time series and of using the time series analysis methods to test for correlations and underlying stochastic noise. Only in the absence of correlation the use of the expression σ/\sqrt{N} is appropriate to characterize the random uncertainty. Otherwise, the Allan variance is an alternate, more appropriate tool to characterize the dispersion of a set of experimental results.

In the case of phase noise, only Modified Allan variance provides accurate results and discriminates flicker from white PM noise. Bretthorst periodogram is a powerful tool of bayesian spectral analysis which detects oscillations in an output signal and identifies peaks at a very close distance. For the case study of mass measurements we consider here, all the above techniques give consistent results.

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